A robust optimization based approach to the general solution of mp-MILP problems

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Abstract

In this work, we focus on the approximate solution of multi-parametric mixed integer linear programming (mp-MILP) problems involving objective function (OFC), left-hand side (LHS) and right-hand side (RHS) uncertainty. A two-step algorithmic procedure is proposed. In the first step a partial immunization against the uncertainty is performed leading to a robust RIM-mp-MILP problem, whereas in the second step explicit optimal solutions of the robust model are derived by applying a decomposition algorithm. Computational studies are presented, demonstrating that (i) the robust RIM-mp-MILP counterpart is less conservative than the conventional robust MILP model, and (ii) the combined robust/multi-parametric procedure is computationally efficient, providing a tight upper bound to the overall global solution of the general mp-MILP problem.

Keywords: multi-parametric programming, robust optimization, mixed-integer linear programming.

1. Introduction

We consider the multi-parametric mixed integer optimization problem (P)

\[ \begin{array}{ll}
\min_{x,y} & (c^T + H\theta)^T x + (d + L\theta)^T y \\
\text{s.t.} & A(\theta)x + E(\theta)y \leq b + F\theta \\
& x \in \mathbb{R}^n, y \in \{0,1\} \in \mathbb{R}^p \\
& \theta \in \Theta = \{ \theta \in \mathbb{R}^q | \theta_1^{\min} \leq \theta_1 \leq \theta_1^{\max}, l = 1, ..., q \},
\end{array} \]

where \( \theta \) denotes the vector of parameters and \( A(\theta) = A^N + \sum_{i=1}^{d} \theta_i A^i \), analogously for \( E(\theta) \). We assume that all matrices and vectors have appropriate dimensions. In the following we will denote by the lower case letter with subscript \( i \), for instance \( [\theta_i] \), the column vector of entries related to the \( i \)-th row of the corresponding matrix.

The presence of uncertainty in mixed integer linear programming models, employed in widespread application fields, including planning/scheduling, hybrid control and process synthesis, significantly increases the complexity and computational effort in retrieving explicit optimal solutions.

Our aim is to find solutions of (P) that (i) are good approximations of the optimal solution and (ii) can be obtained efficiently. In this work, we apply suitable robust optimization techniques to derive solutions of (P). Our approach, denoted as a two-stage method for the solution of general mp-MILP problems, differs from existing methods as we are foremost interested in an immunization against LHS-uncertainty. We formulate a robust counterpart of type RIM-mp-MILP with only OFC- and RHS-uncertainty in the model that closely resembles the parametric nature of the original mp-MILP problem.
problems of this type can be efficiently solved using the algorithm proposed by Faíscu et al. (2009). The method is described next.

2. A two-stage method for the solution of general mp-MILP problems

2.1. The worst-case oriented partially robust counterpart of (P)

The pair \((\bar{x}, \bar{y})^T\) is called a LHS-robust feasible solution of \((P)\) if

\[
\forall \gamma \in \Theta: \quad A^N \bar{x} + E^N \bar{y} + \sum_{i=1}^q \gamma (A_i^x + E_i^y) \leq b + F \theta
\]

for any \(\theta \in \Theta\).

Incorporating (1) into \((P)\) and introducing \(q\) auxiliary variables and additionally \(2q\) linear constraints for each constraint leads to the formulation of the robust counterpart of the general mp-MILP problem. The partially robust counterpart \((RC)\) associated to \((P)\) is given by

\[
\bar{z}(\theta) = \min_{x,y,u_1} ((c + H \theta)^T x + (d + L \theta)^T y)
\]

s.t. \[
[a_i^N]^T x + [e_i^N]^T y + \sum_{i=1}^q (\theta_i^N [a_i]^T x + [e_i]^T y) + \gamma_i u_i \leq [b_i]^T + [f_i]^T \theta, \quad i = 1, \ldots, m
\]

\[
-a_i^N \leq [a_i]^T x + [e_i]^T y \leq u_i, \quad i = 1, \ldots, q, i = 1, \ldots, m
\]

\[
x, y \in R^n, y \in \{0,1\}, u_l \in R^q, \quad i = 1, \ldots, m
\]

\[
\theta \in \Theta \equiv \{\theta \in R^q | \theta_i^\min \leq \theta_i \leq \theta_i^\max, l = 1, \ldots, q\}
\]

where \(r_l := (\theta_i^\max - \theta_i^\min)/2\) denotes the range and \(\theta_i^N := \theta_i^\max - r_l\) the nominal value of \(\theta_i\). The robust model \((RC)\) is a RIM-mp-MILP problem. Every feasible solution of \((RC)\) is a LHS-robust feasible solution of \((P)\). Note that the conventional robust counterpart \((cvRC)\) of \((P)\) corresponds to a fully deterministic MILP problem (Lin et al. (2004)). The solutions of \((cvRC)\) are immune against all data variations in \((P)\). Clearly, every feasible solution of \((cvRC)\) is also feasible for \((RC)\), and consequently for \((P)\).

2.2. A decomposition algorithm for \((RC)\)

We outline the steps of the algorithm presented in Faíscu et al. (2009). The master problem \((M)\) is derived from the RIM-mp-MILP problem \((RC)\) by treating the parameter \(\theta\) as an optimization variable. Due to the bilinear terms in the objective function it corresponds to a nonlinear and non-convex optimization problem. The optimal integer node \(y^\text{opt}\) of \((M)\) is input to \((RC)\), which then results in an mp-LP sub-problem \((S)\). The critical regions of \((S)\), each a subset of \(\Theta\) in which a particular basis remains optimal are uniquely defined by the LP optimality conditions (Gal (1979)). Between every master and sub-problem iteration the MINLP master problem is updated. A new MINLP master problem is solved for each one of the current critical regions. Integer cuts are introduced into the formulation of \((M)\) in order to exclude previously visited integer solutions. Parametric cuts ensure that only integer nodes that are optimal for \((RC)\) for a certain realization of the parameters are considered. The cuts are given by

\[
\sum_{j \in E^k} y^k_j - \sum_{j \in L^k} y^k_j \leq |J^k| - 1, \quad k = 1, \ldots, K,
\]

where \(K\) denotes the number of previously identified integer solutions in this region which have been marked optimal, \(J^k := \{j | y^k_j = 1\}\) and \(L^k := \{j | y^k_j = 0\}\) respectively, and \(|\cdot|\) corresponds to the cardinality, and
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\[(c + H\theta)^T x + (d + L\theta)^T y \leq \bar{z}_k(\theta), \ k = 1, \ldots, K,\]

where \(\bar{z}_k(\theta)\) is the optimal objective value of \((RC)\) at the integer node related to index \(k\). The algorithm terminates in a region where the master problem is infeasible. In order to keep the number of non-convex optimization problems to a minimum, further comparison procedures are omitted. Instead, we retain an envelope of parametric profiles (Dua et al. (2002)) and collect all integer nodes and corresponding continuous solutions that have been identified to be optimal for certain points within a critical region. Function evaluation of the objective values for the parametric profiles stored in the envelope determines the optimal solution of \((RC)\) at any parameter point.

We observe the following properties when the decomposition algorithm is applied to \((RC)\): The critical regions are polyhedral convex. The solutions stored in the envelope of parametric profiles of \((RC)\) are piecewise affine functions.

2.3. Explicit solution of the general mp-MILP problem (P)

The decomposition algorithm outlined in the previous section can be readily extended to address problem \((P)\). If the coefficients of the constraint matrices in \((P)\) are uncertain, the critical regions identified need not be convex. The solutions stored in the envelope of parametric profiles of \((P)\) are piecewise polynomial functions.

The solution of mp-LP sub-problems with LHS-uncertainty is the bottleneck in solving the general mp-MILP problem \((P)\). It either involves enumeration of the parameter space to retrieve the exact solution (Li et al. (2007)), or else an approximation of the solution via global optimization procedures (Dua et al. (2004)). This difficulty is the driving motivation to find a suitable reduction of the model \((P)\) in order to reduce the computational complexity for the decomposition algorithm and to obtain competitive close to optimal solution of \((P)\).

The proposed two-stage method consists of recasting \((P)\) as partially robust RIM-mp-MILP model \((RC)\) as described in Section 2.1, before applying the decomposition algorithm outlined in Section 2.2. An upper bound on the optimal objective value of \((P)\) is obtained.

Note that a lower bound for \((P)\) to serve as a reference value can also be obtained efficiently with the decomposition algorithm by solving a corresponding underestimating problem in which the bilinear terms in the constraints are relaxed using affine functions (McCormick (1976)).

3. Applications of the two-stage method

Example 1. Consider the problem \((P1)\) and its partially robust counterpart \((RC1)\)

\[
\begin{align*}
\max_{x,y} & \quad z(\theta) = \min_{x,y} (\theta_1 x_1 + x_2 + y_1) \\
\text{s.t.} & \quad -x_1 + x_2 + x_3 = \theta_2 + 2y_1 \\
& \quad y_2 - y_1 \leq 0, \ x \geq 0 \\
& \quad y \in \{0,1\}, -5 \leq \theta \leq 5
\end{align*}
\]

\[
\begin{align*}
\max_{x,y} & \quad \bar{z}(\theta) = \min_{x,y} (\theta_1 x_1 + x_2 + y_1) \\
\text{s.t.} & \quad -x_1 + x_2 + x_3 = \theta_2 + 2y_1 \\
& \quad x_1 + 5x_2 + x_4 \leq 1 - 5y_2 \\
& \quad x_1 + 5x_2 - x_4 \leq -1 - 5y_2 \\
& \quad y_2 - y_1 \leq 0, x \geq 0 \\
& \quad y \in \{0,1\}, -5 \leq \theta \leq 5.
\end{align*}
\]

The application of the proposed two-stage method to \((P1)\) required the solution of 9 MINLP and 2 mp-LP problems, returning 6 convex critical regions where an optimal solution exists as depicted in Figure 1.a. In contrast, the decomposition algorithm applied to \((P1)\) required the solution of 13 MINLP and 4 mp-LP problems, computing a total of 9 convex and non-convex critical regions (Figure 1.b). \text{CR}_{\infty} marks a region where \((P1)\) is unbounded. The parametric profiles obtained by the two-stage method provide an upper bound on the exact optimal objective value of \((P1)\), i.e. on the value
related to the best solution among the profiles stored in the envelope with respect to its exact solution for any parameter point. As an example consider \( \theta \in (\text{CR}_1 \cup \text{CR}_5) \cap \text{CR}_\infty \) for which \( z(\theta) = -\infty \), but \( -\infty < \bar{z}(\theta) < +\infty \) holds true.

The conventional robust counterpart of (P1) whose solutions are immunized against LHS-, OFC- and RHS-uncertainty is infeasible for every parameter realization.

Example 2. We consider a sequential scheduling problem with uncertain processing and set-up times (Ryu et al. (2007)). The process consists of two stages with one unit per stage. Three products A, B and C are being processed. The production time of B, denoted by \( \theta_1 \), is unknown but bounded. After two products have been processed at the final stage, this stage may become unavailable for as long as half of the completion time needed for the first two products. The latter variability is modeled as LHS-uncertainty. The objective is to minimize the make-span \( C_{32} \).

The application of the proposed two-stage method required the solution of 3 MINLP problems and 2 mp-LP problems, whereas the decomposition algorithm executed 5 MINLP and 2 mp-LP problems. The parametric profiles derived with both methods are given in Table 1 and Table 2, and the corresponding critical regions are depicted in Figure 2. The optimal make-span obtained with the two-stage method is independent of \( \theta_2 \), the parameter associated to the uncertain set-up time. It yields an overall tighter approximation of the optimal make-span of the original scheduling problem than the optimal make-span of the conventional robust counterpart (Figure 2.b).

<table>
<thead>
<tr>
<th>Critical Region</th>
<th>Optimal Make-span ( C_{32} )</th>
<th>Optimal Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{CR}_1 )</td>
<td>( {3 \leq \theta_1 \leq 5, 0 \leq \theta_2 \leq 0.5 } )</td>
<td>( 1.5\theta_1 + 15 )</td>
</tr>
<tr>
<td>( \text{CR}_2 )</td>
<td>( {3 \leq \theta_1 \leq 8, 0 \leq \theta_2 \leq 0.5 } )</td>
<td>( \theta_1 + 18 )</td>
</tr>
</tbody>
</table>

Table 1: Parametric profiles of Example 2 with the two-stage method

<table>
<thead>
<tr>
<th>Critical Region</th>
<th>Opt. Make-span ( C_{32} )</th>
<th>Opt. Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{CR}_1 )</td>
<td>( {3 \leq \theta_1 \leq 5, 0 \leq \theta_2 \leq \frac{5-\theta_1}{8 + \theta_1} } )</td>
<td>16</td>
</tr>
<tr>
<td>( \text{CR}_2 )</td>
<td>( {4 \leq \theta_1 \leq 8, 0 \leq \theta_2 \leq \frac{5-\theta_1}{8 + \theta_1} } )</td>
<td>( \theta_1 + 18 )</td>
</tr>
<tr>
<td>( \text{CR}_3 )</td>
<td>( 0.08 \leq \theta_2 \leq 0.5, \frac{5-\theta_1}{8 + \theta_1} \leq \theta_2 )</td>
<td>( \theta_1 + 12\theta_2 + 12 )</td>
</tr>
</tbody>
</table>

Table 2: Parametric profiles of Example 2 with the decomposition algorithm
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4. Conclusions

In order to obtain close-to-optimal solutions of the general mp-MILP problem \((P)\), we propose a novel multi-parametric partially robust counterpart of type RIM-mp-MILP which, compared to the original problem, is computationally less expensive to solve with the decomposition algorithm in terms of fewer iterations and avoidance of either discretization of the parameter space or additional global optimization procedures. The second advantage of the proposed two-stage method is the generation of convex critical regions, which significantly simplifies the characterization of the parameter space. Beneficial of the two-stage approach is furthermore the low degree of conservatism of the new robust model compared to the conventional deterministic worst-case robust counterpart. Therefore, we believe the combined robust/parametric optimization approach for general mp-MILP problems to be an attractive alternative to the expensive explicit solution of the original problem and the overly pessimistic results obtained by conventional robust programming.

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References