3 Quantum fields in curved space

Quantum fields in curved space are often used with the expectation that they are a satisfactory approximation to quantum gravity. They introduce many new features that have important physical consequences, the most important so far being black hole evaporation and quantum fluctuations in the early universe.

An important feature with curved space is that the vacuum state can evolve into an excited state. It seems obvious that particles can be produced by a time dependent spacetime background, but particle production can also occur in a locally static spacetime like Schwartzschild.

3.1 Particle states

Particle states are defined with respect to a particular basis of solutions to the free wave equation. These solutions have to be divided in some way into pairs $u_i$ and $\bar{u}_i$ to represent particles and anti-particles. They are also chosen to be be orthogonal, which means that we need to define a product $(,)$ and impose relations

$$(u_i, u_j) = -(\bar{u}_i, \bar{u}_j) = \delta_{ij}, \quad (u_i, \bar{u}_j) = 0.$$  

For the scalar wave equation $(-D^2 + X)\phi = 0$, we use

$$(\phi_1, \phi_2) = -i \int_\Sigma (\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*)\, d\mu$$

where $\Sigma$ is a Cauchy surface with normal derivative $\partial_t$. (A Cauchy surface is a spatial hypersurface that has no edge and every timelike curve intersects it once.) The inner product of any two solutions to the wave equation is independent of the choice of Cauchy surface. We shall assume that there is a time coordinate $t$ which uniquely labels different Cauchy surfaces.

A real free scalar field operator can be decomposed into particle modes

$$\phi = \sum_i \left( a_i u_i + a_i^\dagger \bar{u}_i \right),$$

where the coefficients $a_i$ and $a_i^\dagger$ are anihilation and creation operators which satisfy,

$$[a_i, a_j^\dagger] = \delta_{ij}.$$  

From this it follows that $a_i^\dagger a_i$ has integer eigenvalues which we associate with the number of particles in each mode. The vacuum state is a state which is anihilated by all of the anihilation operators $a_i|0\rangle = 0$.

The choice of vacuum state can be subtle and lead to some interesting phenomena. A given observer may have a preferred choice of basis $u_i$, related to the observer’s time coordinate, which is different from the preferred choice of basis for the physical system. This happens, for example, in de Sitter space, and an observer sees particles in the de Sitter vacuum. If there is an event horizon, then the observer’s set of modes might not even be a complete set of modes for the full spacetime, and then the spacetime vacuum state will generally not be a pure state. This happens for black hole spacetimes, and the result is black hole thermal radiation. Here, we shall only consider field theory effects when the system is in a pure state.
There are a variety of two-point functions to be defined, including the Wightman function $G_W$, the commutator function $G_C$ and the Feynman propagator $G_F$,

\begin{align}
G_W(x, x') &= \langle 0 | \phi(x) \phi(x') | 0 \rangle, \\
G_C(x, x') &= \langle 0 | [\phi(x), \phi(x')] | 0 \rangle, \\
G_F(x, x') &= \langle 0 | T \phi(x) \phi(x') | 0 \rangle.
\end{align}

The Feynman propagator depends on a time-ordering operation $T$, which places the operator with the larger value of $t$ to the left. There is a causality condition that the commutator function vanishes, and $G_F = G_W$, for spacelike separated points. The limit of small spacelike separation can be obtained analytically using a WKB type of approximation, usually referred to in this context as the Hademard expansion. For small spacelike separation,

\begin{align}
G_W(x, x') = U(x, x') \sigma^2 - d V(x, x') \ln(\sigma) + W(x, x').
\end{align}

where $\sigma$ is the geodesic distance between $x$ and $x'$. The functions $V$ and $W$ are finite at $x = x'$, and can be determined order by order as expansions in powers of $\sigma^2$ using the green function equation.

The two-point functions also have mode expansions, obtained by using (3), for example

\begin{align}
G_W(x, x') = \sum_i u_i(x) \bar{u}_i(x').
\end{align}

It has become fashionable to consider effective field theories which are valid only in a low energy regime. If the modes are ranked with larger index $i$ for larger energy (frequency), then the mode sum should be truncated at a maximum value of $i$. Alternatively, since the unbounded values of $i$ are needed to produce the divergent terms in the small distance expansion (8), the effective theory can be obtained in some cases by omitting the divergent terms from the Hademard expansion. This procedure provides a natural way to regularise some expressions which would otherwise come out to be infinite.

### 3.2 Effective action

The effective action is an important tool used to study of interacting field theories. Consider the case of a scalar quantum field on a spacetime with metric $g$. We require that $g$ has a well-defined time coordinate so that time-ordering can be defined. The generating function for time-ordered green functions for the scalar field theory with an external current $J$ is given in path integral form by

\begin{align}
Z[g, J] = \int d\mu[\phi'] \psi_f(\phi')^\dagger \psi_i(\phi') e^{iS_J[\phi']}.
\end{align}

where the action $S_J$ includes a linear coupling to the source $J$. The field has initial and final states $\psi_i$ and $\psi_f$. If particle production is significant, then the ‘closed time path’ generating functional, similar to the one used for thermal field theory, should be used. However, we shall focus on simple situations where $\psi_i$ and $\psi_f$ are approximate vacuum states and particle production can be ignored.

The field and energy momentum expectation values for the given states are given by

\begin{align}
\phi = -\frac{i}{\int Z} \frac{\delta Z}{\delta J}, \quad T^{ab} = -\frac{1}{2} \frac{\delta Z}{\delta g_{ab}}.
\end{align}

Assuming the first equation can be inverted to give $J \equiv J[\phi]$ allows the definition of an effective action $\Gamma \equiv \Gamma[g, \phi]$ by

\begin{align}
\Gamma[g, \phi] = -i \ln Z + \int_M J \phi
\end{align}

The definition leads to the equations

\begin{align}
\frac{\delta \Gamma}{\delta \phi} = J, \quad \frac{\delta \Gamma}{\delta g_{ab}} = -\frac{1}{2} T^{ab}.
\end{align}
In perturbation theory, the effective action of a scalar field is given by

\[ \Gamma[\phi] = S[\phi] + \frac{1}{2} \log \det(\Delta / \mu_R^2) + \ldots, \tag{14} \]

where \( \Delta \) is the operator obtained from a second order variation of the action about the fields \( \phi \) and \( \mu_R \) is an arbitrary mass scale. For example, for scalar field theory with Lagrangian

\[ L = -\frac{1}{2}(\nabla^2 \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4, \tag{15} \]

the operator \( \Delta = -\nabla^2 + \mu^2 + 3\lambda \phi^2 \).

### 3.3 \( \zeta \)-function regularisation

The \( \zeta \)-function provides a convenient method to define the determinant in the one loop contribution to the effective action,

\[ W = \frac{1}{2} \log \det(\Delta / \mu_R^2). \tag{16} \]

By performing an analytic continuation of the metric it is possible to turn \( \Delta \) into an elliptic operator, meaning that it is possible to write \( \Delta = g^{ab} \partial_a \partial_b + \ldots \), where \( g^{ab} \) is a positive definite metric and the dots indicate lower order terms. By grouping terms together we can express \( \Delta = -D^2 + X \), where \( D \) is a gauge covariant derivative.

The \( \zeta \)-function for an elliptic operator is defined by a functional trace \(^1\)

\[ \zeta(s, \Delta) = \text{tr} \left( \Delta^{-s} \right). \tag{17} \]

If the operator has a discrete set of eigenvalues \( \lambda_n \), this is equivalent to

\[ \zeta(s, \Delta) = \sum_{n=1}^{\infty} \lambda_n^{-s}. \tag{18} \]

For second order operators, the sum only converges when \( 2s \) is larger than the number of dimensions \( d \), but we remove this restriction by analytic continuation to values of \( s \) in the complex plane. A crucial property of the \( \zeta \)-function is that the analytic continuation is regular at \( s = 0 \).

Now we are ready to define the operator determinant by

\[ \log \det(\Delta / \mu_R^2) = -\zeta'(0, \Delta) - \zeta(0, \Delta) \log \mu_R^2. \tag{19} \]

This formula would simply be an identity if the number of eigenvalues was finite. Analytic continuation enables us to calculate the one loop effective action explicitly in a wide range of examples, including spheres, balls, shells and so on.

### 3.4 Heat kernel coefficients

It will prove convenient to define a smeared version of the \( \zeta \)-function by

\[ \zeta(s, \Delta, \omega) = \text{tr} \left( \Delta^{-s} \omega \right). \tag{20} \]

The analytic properties of the \( \zeta \)-function follow from the integral representation based on (18),

\[ \zeta(s, \Delta, \omega) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr} \left( \omega e^{-t\Delta} \right) dt \tag{21} \]

---

\(^1\)In Dirac notation, the functional trace \( \text{tr}(A) = \sum \langle n | A | n \rangle \), where \( | n \rangle \) is any basis of functions.
The kernel $e^{-\Delta t}$ satisfies the heat equation, and can be analysed using asymptotic approximations in $t$, which is closely related to the asymptotic approximations of the green function in powers of $\sigma$ (8). It can be shown rigorously that, for a second order operator, the trace has an asymptotic expansion

$$\text{tr} \left( \omega e^{-\Delta t} \right) \sim t^{-d/2} \sum_{n=0}^{\infty} B_n[\Delta, \omega] t^{n/2} \quad (22)$$

Furthermore, the heat kernel coefficients $B_n[\Delta, \omega]$ for $\Delta = -D^2 + X$ only depend on local combinations of covariant tensors, such as curvatures and covariant derivatives of $X$.

When the integral converges, (21) can be written in terms of a Hankel contour $C$,

$$\zeta(s, \Delta, \omega) = (-1)^s \frac{\Gamma(1-s)}{2\pi i} \int_C t^{s-1} \text{tr} \left( \omega e^{-\Delta t} \right) dt \quad (23)$$

which provides an analytic extension to $s < d/2$. In particular,

$$\zeta(0, \Delta, \omega) = B_d[\Delta, \omega] \quad (24)$$

and there are poles in the $\zeta$-function at $s = (d-N)/2$ for $N < d$.

The properties of the heat kernel coefficients under conformal transformations often play an important role in their evaluation. We introduce a new metric by a Weyl rescaling, $g_\epsilon = e^{2\omega} g$, and a new function $X_\epsilon$. These can be used in order to express the rescaled operator $e^{-2\omega \Delta}$ in '−$D^2 + X\epsilon$' form,

$$e^{-2\omega \Delta} = e^{(d/2-1)\omega} \left( -D_\epsilon^2 + X_\epsilon \right) e^{-(d/2-1)\omega} \quad (25)$$

The exponential factors on the right cancel when evaluating the determinant, and so the heat kernel coefficients of $e^{-2\omega \Delta}$ only depend on the curvature of $g_\epsilon$ and covariant derivatives of $X_\epsilon$. Table 3.6 implies that

$$X_\epsilon = e^{-2\omega} \left( X - \frac{1}{2} (d-2) \epsilon \omega_{\alpha} \omega^\alpha - \frac{1}{4} (d-2)^2 \epsilon^2 \omega_\alpha \omega^\alpha \right) \quad (26)$$

(In a conformally invariant theory, the transformation of $X$ to $X_\epsilon$ is automatic. In other cases the transformation is a technical device). Differentiating the functional trace (17) gives,

$$\partial_\epsilon \zeta(s, \Delta_\epsilon) \big|_{\epsilon = 0} = 2s \text{tr} (\omega \Delta^{-s}) = 2s \zeta(s, \Delta, \omega) \quad (27)$$

From the residues at $s = (d-N)/2$,

$$\partial_\epsilon B_N(\Delta_\epsilon) \big|_{\epsilon = 0} = (d-N) B_N[\Delta, \omega]. \quad (28)$$

The first thing we notice is that $B_N[\Delta, \omega]$ can always be recovered from $B_N(\Delta)$ by a conformal transformation. Secondly, we see that $B_N(\Delta)$ is conformally invariant in $N$ dimensions, and must be composed of conformally invariant tensors or total derivatives.

Some examples of heat kernel coefficients are tabulated, where $B_N$ (≡ $B_N[\Delta, 1]$) is written in the form

$$B_N = \frac{1}{(4\pi)^{d/2}} \int_M b_N + \frac{1}{(4\pi)^{d/2}} \int_{\partial M} c_N. \quad (29)$$

The total derivatives in the tables are related to the Gauss-Bonnet identity in $d$ (even) dimensions,

$$\chi = \frac{1}{(4\pi)^{d/2}} \frac{1}{(d/2)!} \left( \int_M q_d + \int_{\partial M} q_{d-1} \right) \quad (30)$$

The quantity $\chi$ is the Euler number, which is always an integer. For an explicit expression, $q_d = 2^{-d/2} d! R_{\mu\nu\cdots} R_{\rho\sigma} \cdots R_{\tau\sigma}$. For example, $q_2 = R$. 

4
The conform variation of the effective action is itself a useful quantity. Consider the one loop effective action for the conformally rescaled theory, as defined above:

\[ W[\omega] = \frac{1}{2} \log \det(e^{2\omega} \Delta / \mu^2) \] (31)

The transformation of the \( \zeta \)-function (27) implies

\[ \partial_\epsilon W[\epsilon, \omega] = -\zeta(0, \Delta, \omega) = -B_d[\Delta, \omega] \] (32)

A conformally invariant theory is one in which the operators simply rescale under a conformal transformation. However, the effective action can still change under the conformal transformation. In this case, the trace of the stress-energy tensor is called the conformal anomaly. It is given by

\[ \langle T^a_a \rangle = -2g^{ab} \delta \Gamma \delta g_{ab} = -d \partial_\epsilon W[\epsilon, \omega] = dB_d[\Delta, \omega] \] (33)

The conformal anomaly can be put to a practical use. By integrating over \( \epsilon \), it is possible to relate the effective action for two different backgrounds. The integration produces a new expression,

\[ W[\omega] - W[0] = C[\Delta, \omega] \] (34)

where \( C[\Delta, \omega] \) is called the cocycle function and can be expressed in terms of local invariants. For example, if \( d = 2 \) and \( \Delta = -\nabla^2 + X \), Table 2 gives

\[ \zeta(0, \Delta, \omega) = \frac{1}{4\pi} \int_M \left( \frac{1}{6} R - X \right) \omega \, d\mu_\epsilon + \frac{1}{4\pi} \int_{\partial M} \left( \frac{1}{3} \theta - \frac{1}{2} \omega_n \right) \omega \, d\mu_\epsilon \] (35)

After integration over \( \epsilon \),

\[ C = \frac{1}{4\pi} \int_M \left\{ (X - \frac{1}{2} R) \omega - \frac{1}{6} (\nabla \omega)^2 \right\} \, d\mu \\
+ \frac{1}{4\pi} \int_{\partial M} \left\{ -\frac{1}{3} \theta \omega - \frac{1}{2} \omega_n \right\} \omega \, d\mu \] (36)

Expressions such as this enable us to find \( W[\omega] \) once \( W[0] \) is known.

### 3.5 Quantum field theory in de Sitter space

De Sitter space is an important example of a curved spacetime in which many quantities can be calculated explicitly due to the high degree of symmetry. Four dimensional de Sitter space is a cosmological model with constant expansion rate, of particular relevance to the inflationary scenario.

De Sitter space can be obtained as a \( d \) dimensional surface in \( d + 1 \) dimensional Minkowski space,

\[ \eta_{ab} X^a X^b = r^2 \] (37)

where \( X^0, \ldots, X^d \) are the Minkowski coordinates and \( \eta_{ab} = \text{diag}(-1, 1, \ldots, 1) \). De Sitter space is invariant under the Lorentz group \( SO(d, 1) \). The symmetry is not necessarily a symmetry of the quantum theory, but the \( SO(d, 1) \) invariant quantum field theory seems the most natural.

We take a free massive scalar field in our dimensions and start with the spatially flat coordinate system

\[ ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2), \] (38)

where \( H = 1/r \) is the expansion rate. The relevant modes are \( u_k(t) e^{-k \cdot x} \), and the wave equation becomes

\[ \frac{d^2 u_k}{dt^2} + 3H \frac{du_k}{dt} + (k^2 e^{-2Ht} + m^2) u_k = 0. \] (39)
One set of normalised solutions of special interest are
\[ u_k = \frac{\sqrt{\pi}}{2} H_\nu(kr e^{-Ht})H^{1/2}e^{-3Ht/2}, \]  
where \( H_\nu \) is a Hankel function with index \( \nu^2 = 9/4 - r^2m^2 \). The small mass limit \( \nu = 3/2 \) is widely used in the discussion of fluctuations in the inflationary universe scenario. The equal-time spatial Fourier transform of the Wightman function (9) in that case defines a power spectrum,
\[ u_k^*(t)u_k(t) = \frac{H^2}{2k^3} + \frac{e^{-2Ht}}{k}. \]

For long physical wavelengths, \( k << H e^{Ht} \), this depends only on \( k \) and leads to the ‘scale-free’ spectrum of density perturbations which is one of the most important predictions of inflationary theory.

The \( SO(d,1) \) symmetry of the particle propagators is not explicit from this construction. The symmetry can be made more explicit by constructing the propagators on a Euclidean manifold and using analytic continuation back to de Sitter space. We start by replacing the coordinate \( X \), placed in the Euclidean manifold and using the symmetry can be made more explicit by constructing the propagators on a Euclidean manifold and using analytic continuation back to de Sitter space. We start by replacing the coordinate \( X \) and letting \( X \rightarrow \chi \) and the equation becomes
\[ -\sin^{1-d} \chi (\sin^{d-1} \chi G')' + r^2m^2 = 0. \]

As \( x \rightarrow x' \), the equivalence principle implies that the Green function approaches the flat space result,
\[ G \rightarrow \omega_{d-1}(r \chi)^{2-d} \]
where \( \omega_d \) is the volume of a \( d \) sphere. The solution to (43) and (44) in four dimensions is
\[ G = A_4 F(\frac{3}{2} + \nu, \frac{3}{2} - \nu; 2; \cos^2(\chi/2))r^{-2} \]
where \( \nu^2 = 9/4 - r^2m^2 \). The constant \( A_4 = \Gamma(3/2 + \nu)\Gamma(3/2 - \nu)/(32\pi^2) \) can be obtained from a small \( \chi \) approximation to the Green function using Abramowitz and Stegun formula 15.3.12. With this value of \( A_4 \), the hypergeometric function identity gives a Hadamard-like expression for the Green function,
\[ G = \frac{1}{16\pi^2r^2} \left\{ \cos^2\frac{\chi}{2} + \left( 1 - \nu^2 \right) \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(a)\Gamma(b)n!(n + 1)!} \times \sin^{2n}\frac{\chi}{2} \left[ \ln \sin^2\frac{\chi}{2} + \psi(a + n) + \psi(b + n) - \psi(n + 2) - \psi(n + 1) \right] \right\} \]

where \( a = 3/2 + \nu \) and \( b = 3/2 - \nu \) and \( \psi(z) \) is the digamma function.

For quantum theory in de Sitter space we apply an analytic continuation of the green function back to the de Sitter spacetime, where it becomes the Feynman propagator \( G_F \). The analytic continuation can be done by re-introducing the Minkowski coordinates,
\[ \cos^2\left(\chi/2\right) = 1 - \left((X - X')/2r\right)^2, \]
and letting \( X^0 \rightarrow -iX^0 \). In the closed universe cosmological coordinates, \( X^0 = r \sinh(t/r) \) and the green function is periodic under \( t \rightarrow t + 2\pi i r \), an indication of thermal behaviour with temperature \( 1/(2\pi r) \).

The one loop contribution to the effective action can be evaluated from the green function using a simple trick, by differentiating Eq. (16) with respect to the mass,
\[ \partial_{m^2}W = \frac{1}{2} \text{reg} \, \text{tr}(\Delta^{-1}\partial_{m^2}\Delta) = \frac{1}{2} \text{reg} \, \text{tr}(G), \]
where the ‘reg’ denotes the removal of the terms in the green function which diverge as \( x \to x' \). This can be done using Eq. (46), and the one-loop effective potential \( V_1 \) is given by,

\[
V_1 = -\frac{1}{32\pi^2 r^2} \int (\nu^2 m^2 - 2) \left[ -\log(4\nu^2) + \psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) - (1 - \psi(2)) \right] dm^2 - \frac{1}{2} B_4 \ln \mu_R^2. \quad (49)
\]

The Ricci curvature scalar \( R = 12/r^2 \). For small curvature \( R << m^2 \),

\[
V_1 \approx \frac{m^4}{64\pi^2} \left( \ln \frac{m^2}{\mu_R^2} - \frac{3}{2} \right) - \frac{m^2 R}{192\pi^2} \left( \ln \frac{m^2}{\mu_R^2} - \frac{4}{3} \right) + \frac{29R^2}{35460\pi^2} \ln \frac{m^2}{\mu_R^2} + \ldots \quad (50)
\]

This result agrees with the flat space effective potential in the limit \( R \to 0 \). Furthermore, the logarithmic terms from the integral and the \( B_4 \) coefficient calculated from Table 2 agree. The large curvature limit is

\[
V_1 \approx -\frac{R^2}{2160\pi^2} \ln \frac{R}{\mu_R^2} + \frac{R^2}{768\pi^2} \ln \frac{m^2}{\mu_R^2} - \frac{m^2 R}{192\pi^2} \left( \ln \frac{R}{48\mu_R^2} + \frac{7}{3} \right) + \ldots \quad (51)
\]

The first term is a constant of integration in (49) which has been recovered from the \( B_4 \) term. The second term diverges in the massless limit, raising an issue which is discussed further below. The third term can be compared to \( m^2 T^2 \) terms which arise in the effective potential at high temperature in flat space, but the sign here gives a destabilising effect.

The effective action of a free scalar field of mass \( m \) can also be calculated by direct evaluation of the \( \zeta \)–function on the \( d \) dimensional sphere. The eigenvalues are \( \lambda = (l + d - 1)r^{-2} + m^2 \) with degeneracy

\[
d_l = (2l + d - 1) \Gamma(l + d - 1) \Gamma(d) \Gamma(l + 1). \quad (52)
\]

This direct approach is quite complicated (see Dowker 1994). For \( m = 0 \) and \( d = 4 \) (when the \( l = 0 \) mode is excluded), the effective potential can be expressed in terms of Riemann \( \zeta \)–functions,

\[
V = -\frac{1}{4\pi^2 r^2} \zeta'(0) = -\frac{1}{4\pi^2 r^2} \left( \frac{2}{3} \zeta_R(-3) + \frac{13}{3} \zeta_R(-1) - \frac{15}{16} \ln 3 \right).
\]

This fixes the constant of integration in (49).

The green function evaluated at fixed cosmological time gives a measure of the spatial fluctuations in inflation. For \( r^2 \cosh^2(t/r) \gg (X - X')^2 \gg r^2 \), the cosmological distance \( |x - x'| \approx |X - X'| \). Using Abramowitz and Stegun formula 15.3.8 gives

\[
G_F \to \frac{\Gamma\left(\frac{3}{2} - \nu\right) \Gamma(2\nu)}{16\pi^2 \Gamma\left(\frac{1}{2} + \nu\right)} H^2 \frac{1}{2} (H|x - x'|)^{2\nu - 3}. \quad (54)
\]

If the mass is small,

\[
G_F \to \frac{3H^4}{8\pi^2 m^2} - \frac{H^2}{8\pi^2} \log(H|x - x'|). \quad (55)
\]

The spatial Fourier transform in the range \( H << k << He^{Ht} \) produces the same \( H^2 k^{-3} \) behaviour which was obtained from the mode expansion.

The first term in Eq. (55) suggests that there are problems in the massless limit. Small, or even negative values of \( m^2 \) are used in inflationary models and so we need to deal with these situations. We begin by noticing that the problematic term is the first term of an eigenmode expansion for the Euclidean Green function,

\[
G = \sum_{l=0}^{\infty} \left( \frac{U_l(x)U_l^*(x')}{m^2 + l(l + 2)H^2} \right). \quad (56)
\]

where \( U_n \) are volume-normalised eigenmodes. We introduce a new propagator which breaks the de Sitter invariance and splits into two terms,

\[
G_F = G_0 + G_1, \quad (57)
\]
where $G_1$ is obtained by analytic continuation of the Green function with the $l = 0$ mode removed.

Starting from (43), the equation for $G_1$ implies the equation for $G_0$

$$(-\nabla^2 + m^2)G_1 = -i\delta - \frac{3H^4}{8\pi^2}$$

$$(-\nabla^2 + m^2)G_0 = \frac{3H^4}{8\pi^2}$$

We assume $G_0 \equiv G_0(t + t')$ in the flat space coordinate system. We also impose a condition that $G_0$ and $\partial_t G_0$ are both bounded when $t = t_0 = 0$ and $m \to 0$. The solution is then

$$G_0 \approx \begin{cases} 
\frac{3H^4}{8\pi^2 m^2} \left(1 - e^{-m^2(t+t')/3H}\right) & m^2 \neq 0 \\
\frac{H^3}{8\pi^2} (t + t') & m^2 = 0
\end{cases}$$

This growth in the green function is similar to the spread of a gaussian wave packet in quantum mechanics. When $m^2 < 0$, the exponential growth matches the classical behaviour of the background scalar field rolling down a potential hill.
### 3.6 Tables

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{ab][e^{2\omega}g]$</td>
<td>$R_{ab} + (d - 2)(\omega_{ab} - \omega^{c} \omega_{cb}) + (\omega^{c} + (d - 2)\omega^{c e} g_{ab})$</td>
</tr>
<tr>
<td>$R[e^{2\omega}g]$</td>
<td>$e^{-2\omega}(R + 2(d - 1)\omega^{a} + (d - 1)(d - 2)\omega^{a e} g_{ab})$</td>
</tr>
<tr>
<td>$\nabla^{2}[e^{2\omega}g]$</td>
<td>$e^{-2\omega}(e^{-(d/2 - 1)\omega} - \frac{1}{2}(d - 2)\omega^{a} - \frac{1}{4}(d - 2)^{2}\omega^{a e} g_{ab})$</td>
</tr>
</tbody>
</table>

Table 1: Behaviour of the curvature under Weyl rescaling.

<table>
<thead>
<tr>
<th>Term</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{0}$</td>
<td>$\text{tr}(1)$</td>
</tr>
<tr>
<td>$b_{2}$</td>
<td>$\text{tr} \left( \frac{1}{2} q_{2} - X \right)$</td>
</tr>
<tr>
<td>$b_{4}$</td>
<td>$\text{tr} \left( -\frac{1}{360} q_{4} + \frac{1}{120} C^{2} + \frac{1}{2}(X - \frac{1}{5} R)^{2} + \frac{1}{12} F^{ab} F_{ab} \right)$</td>
</tr>
<tr>
<td>$b_{6}$</td>
<td>$\text{tr} \left( \frac{64}{27} \cdot 27 \cdot 7! q_{6} + \frac{496}{27} \cdot 7! C_{a}^{c} + \frac{4}{27} C_{a}^{d} C_{d}^{c} - \frac{1}{2} C^{2} R \right)$</td>
</tr>
<tr>
<td>&amp; $+ \frac{1}{270} C^{2} (X - \frac{4}{5} R) - \frac{1}{6} (X - \frac{1}{5} R)^{3} - \frac{1}{12} ((\nabla C)^{2} - \frac{1}{5} C^{2} R)$</td>
<td></td>
</tr>
<tr>
<td>&amp; $- \frac{1}{180} (\nabla F)^{2} - \frac{1}{30} F^{3} - \frac{1}{60} C_{abcd} F_{ac} F_{bd} - \frac{1}{12} (X - \frac{1}{5} R) F^{2}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Heat kernel coefficients $b_{N}$ for the operator $-D^{2} + X$ on a manifold in $d$ dimensions. $R$ is the Ricci scalar of the manifold and $C$ is the a combination of curvature components which equals the Weyl tensor when $d = N$, $C_{abcd} = R_{abcd} - \frac{1}{N-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(N-1)(N-2)} R_{a[c}g_{d]b}$. Metric contractions in $C^{2} = C_{abcd} C^{abcd}$, $C^{3} = C_{a}^{d} C_{b}^{c} C_{c}^{f} C_{d}^{e}$ and $C_{3} = C_{a}^{c d} C_{c d}^{e f} C_{e f}^{a b}$ are also evaluated in $N$ dimensions.
Table 3: Boundary heat kernel coefficients $c_N$. $k_{ab}$ is the extrinsic curvature, $\theta$ its trace and $\sigma_{ab} = k_{ab} - \theta/(N-1)$. The boundary conditions are part Dirichlet $P_-\psi = 0$ and part Robin $(\psi + \partial_n)P_+\psi = 0$.

References


