Quantum Mechanics on Moduli Spaces II

Moduli space—a parameter space of static, finite energy solutions to a classical field theory.

The aim is to construct a quantum theory of these parameters. This will be done within a general framework that can be applied equally well to the scattering of BPS monopoles or to molecular spectra.

This talk will concentrate on the vacuum energy.

Plan

1. The CP(1) model in 2+1 dimensions
2. Factor ordering problems
3. The Reduced Hamiltonian
4. Vacuum Energies
5. Geometrical Effects
6. Conclusion
The CP(1) model in 2+1 dimensions

The model: A complex scalar field \( u(z, t) \), with

\[
T = \frac{1}{g^2} \int (1 + |u|^2)^{-2} |\dot{u}|^2 \, d^2 z \\
V = \frac{1}{g^2} \int (1 + |u|^2)^{-2} (\partial_\mu u)^*(\partial^\mu u) \, d^2 z
\]

where \( g \) is a coupling constant.

The soliton solutions are characterised by a topological index \( n \) and energy \( 2\pi n/g \). For example, the two soliton solution

\[
u = \frac{z_1}{z^2 + z_2} \quad \text{moduli } z_1, z_2
\]

Slowly moving solitons have \( z_i \equiv z_i(t) \). The induced geometry is obtained by substituting into \( T \),

\[
T = \frac{1}{2} g_{i\bar{j}} \dot{z}^i \dot{z}^{\bar{j}}
\]

The metric \( g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Omega(|z_1|, |z_2|) \), where \( \Omega \) is an elliptic integral.
Moduli spaces

Bogonol’nyi energy bounds also exist for Kähler sigma models and for BPS monopoles.

*Moduli space*—the parameter space of static, finite energy solutions.

In the slow-motion approximation

\[
\begin{align*}
\text{Classical dynamics on } \mathcal{M} & \equiv \text{classical soliton motion} \\
\text{Quantum dynamics on } \mathcal{M} & \equiv \text{quantised soliton motion}
\end{align*}
\]

There is also an extension to “approximate” moduli spaces which have a potential on \(\mathcal{M}\).

AIM: Describe the reduction of quantum mechanics from \(M\) to \(\mathcal{M}\).
Quantisation Ambiguities

\[ H = \frac{1}{2} g^{ab}(x)p_ap_b + V(x) \]

Quantise by \([x^a, p_b] = i\hbar \delta^a_b\). Where does the \(g^{ab}\) go?

Use the covariant Laplacian (Podolski 1928):

\[ H = -\frac{1}{2} \hbar^2 \Delta + V \]

We can always add more terms,

\[ i\xi \hbar R^a_b [x^a, p_b] \]

which produces a \(\xi \hbar^2 R\) term.

Similar ambiguities exist in the definition of the path integral (DeWitt 1957).
More Symmetry

(1) Riemannian Symmetric spaces ($\nabla R = 0$)

$$H = -\frac{1}{2} (\nabla + A)^2 + X$$

The symmetry implies $A = 0$ and $\nabla X = 0$. (e.g. symmetric top)

(2) Supersymmetry (Alfaro et al.)

$$H = \{Q, Q^*\}$$

The wave function $\psi(x, \lambda^i) = \psi(x) + \psi_i(x) \lambda^i + \ldots$, then

$$-\frac{\hbar^2}{2} \Delta \psi = E\psi \quad \text{p-forms}$$

The factor ordering is “fixed” but the result is still ambiguous.
**Born-Oppenheimer reduction**

The quantisation of the moduli space parameters is induced by the larger theory into which the parameters are embedded.

\[
\begin{array}{c@{}c@{}c@{}c}
\text{classical F.T.} & \rightarrow & \text{truncate} \\
\downarrow & \downarrow & \downarrow \\
\text{quantum F.T.} & \rightarrow & \text{quantum mechanics}
\end{array}
\]

The reduction of quantum theory to the manifold \( \mathcal{M} \) is an example of Born–Oppenheimer approximation. We integrate out the modes of vibration about the moduli space.

Begin with a finite system of variables \( x^\mu \),

\[
L = \frac{1}{2} \delta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V(x)
\]

The moduli space \( \mathcal{M} \) is a collection of minima of \( V \), coordinates \( x^i \).

Vibrational frequencies \( \omega_I^2 \) are eigenvalues of the hessian \( \partial_I \partial_J V \). Focus on interactions of energy \( \ll \) smallest \( \omega_I \). Use a Born-Oppenheimer ansatz

\[
\Psi(x^\mu) = \sum_n \psi(x^i) f_n(x^I, x^i)
\]

with degenerate perturbation theory.
The Basic Result

\[ \left( -\frac{\hbar^2}{2} \Delta + V(x^i) + \frac{\hbar^2}{4} R - \frac{\hbar^2}{8} K^2 + O(\epsilon) \right) \psi = E\psi \]

where \( K \) is the traced extrinsic curvature and

\[ V = V(x)|_\mathcal{M} + \frac{\hbar}{2} tr(\omega) + \frac{\hbar^2}{16} tr(\omega^{-1} \nabla \omega)^2 + \ldots \]

Notes:

(1) The curvature terms were first published by Maraner 1996.
(2) The vibrational frequencies \( \omega_I^2 \) for a soliton are obtained from classical perturbations of the soliton.
(3) The correction to the vacuum energy is new. Also:

\[ \frac{1}{32} \sum_{IJ} \omega_I^{-1} \omega_J^{-1} V_{IJII} - \frac{1}{48} \sum_{IJK} (\omega_I + \omega_J + \omega_K)^{-1} \]

\[ \omega_I^{-1} \omega_J^{-1} \omega_K^{-1} V_{IJK}^2 - \frac{1}{32} \sum_{IJK} \omega_I^{-2} \omega_J^{-1} \omega_K^{-1} V_{IJJ} V_{IKK} \]

(4) Can be derived from a path integral.
(5) Can be generalised to cases where the original manifold is curved and there are fermions.
The Vacuum Energy

How do we calculate the casimir force between two CP(1) solitons?

(1) Place the solitons on a torus and use variational methods to get the eigenvalues. (Shiiki 1999)
Phase Shifts

\[
\frac{1}{2} tr(\omega) = -\frac{1}{2\pi} \int_0^\infty dk \sum_{m=-\infty}^{\infty} \delta_m(k) \quad (\text{Schwinger 1954}).
\]

Subtracting Born approximations removes the ultra violet divergences (Jaffe 1998).

For Zeta function regularisation, the heat kernel

\[
K(t) = K_0(t) + \frac{2}{\pi} \int_0^\infty dk \sum_r \delta_r(t) \quad (\gamma) \quad (\text{Jaffe 1977}).
\]

Compare the small time expansion

\[
K(t) \sim t^{-d/2} \sum_{n=0} B_n t^n.
\]

The first Born approximation produces \( B_1 \), etc.

Single soliton of width \( \alpha \), \( \delta_m \equiv \delta_m(k\alpha) \) and the vacuum energy \( \approx 0.25 \alpha^{-1} \).
Quantum scattering of solitons

\[
\left( -\frac{\hbar^2}{2} \Delta + \frac{\hbar^2}{2} tr(\omega) + \frac{\hbar^2}{4} R - \frac{\hbar^2}{8} K^2 \right) \psi = E \psi
\]

(1) \(tr(\omega)\): Solve numerically.

(2) \(K = 0\) for a Kahler hypersurface embedded in a kahler manifold.

(3) \(R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log |g|\). Width \(w \ll \text{separation} \ d\),

\[
R \sim \frac{2}{\pi w^2} \left( \log \frac{4d}{w} \right)^{-3}
\]

(4) The geometrical effects induce a potential \(\hbar^2 R/2\) on the moduli space.
Conclusions

(1) The Born-Oppenheimer reduction from $M$ to the moduli space $\mathcal{M}$ produces

$$\left(-\frac{\hbar^2}{2}\triangle + V(x^i) + \frac{\hbar^2}{4}R - \frac{\hbar^2}{8}K^2 + O(\epsilon)\right)\psi = E\psi$$

where $K$ is the traced extrinsic curvature and

$$V = V(x)_{|\mathcal{M}} + \frac{\hbar}{2}tr(\omega) + \frac{\hbar^2}{16}tr(\omega^{-1}\nabla\omega)^2 + \ldots$$

It includes the effects of mixing between the ground state and excited states of the soliton.

(2) The vacuum energy can be calculated numerically using variational methods to get the eigenvalues if the spectrum is discrete. Edge effects are a serious problem.

(3) The vacuum energy can be calculated numerically using phase shifts if the spectrum is continuous. For a single $\text{CP}(1)$ soliton case, $\frac{1}{2}tr(\omega) \propto \alpha^{-1}$ and it is negative.